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1987 J. Phys. A: Math. Gen. 20 2405

(http://iopscience.iop.org/0305-4470/20/9/027)

J. Phys. A: Math. Gen. **20** (1987) 2405–2413. Printed in the UK

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Received 11 June 1986, in final form 6 October 1986

Abstract. The equation of motion of an electron undergoing cyclotron emission in a uniform magnetic field is considered. A solution for the orbit is obtained in a fairly simple form for relativistic velocities. The conflict of basic conservation laws, as noted previously, is no longer evident in the present treatment. Difficulties in earlier works have arisen because the deviation of the electron orbit from circularity was not estimated properly. As particular consequences of the more complete theory, it is shown that (i) when the fractional energy loss per cycle approaches unity there are significant changes in the particle orbit, and (ii) when the fractional energy loss per pulse (or beam) crossing approaches unity, the radiation characteristics also exhibit modifications.

1. Introduction

The conflict of basic conservation laws in cyclotron radiation theory was addressed in previous literature (Lieu *et al* 1983, Das Gupta 1984). In a recent work (Lieu *et al* 1987) we suggest that the controversy can be settled by a careful consideration of the balance of energy, momentum and angular momentum when the two-body system of an electron and a source of magnetic field emits a photon. The idea was hinted at in an earlier paper (White and Parle 1985) which studies the effects of recoil on the electron orbit. At present we explore some of the consequences of radiation reaction in cyclotron physics. The mathematical computations assume particularly simple forms in non-relativistic and ultra-relativistic limits. Our concern is in the latter, because little observable significance is expected for low electron velocities.

2. Equation of motion and its solution

The equation of motion of an electron in a homogeneous and static magnetic field is given, in 4-vector notation, by

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} = F_\mu^{\text{field}} + F_\mu^{\text{rad}}$$

where

$$F_\mu^{\text{field}} = (0, \gamma \boldsymbol{v} \times \boldsymbol{B})$$

$$F_\mu^{\text{rad}} = \frac{2e^2}{3mc^3}\left[\frac{\mathrm{d}^2 p_\mu}{\mathrm{d}\tau^2} + \frac{p_\mu}{m^2c^2}\left(\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau}\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau}\right)\right]$$

(see Jackson 1975, Rohrlich 1965). It is sufficient to work exclusively on the three spacelike components of this equation, namely

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}=e\boldsymbol{v}\times\boldsymbol{B}+\frac{2e^2}{3mc^3}\left\{\left(\frac{1}{mc^2}\frac{\mathrm{d}E}{\mathrm{d}t}\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}+\gamma\frac{\mathrm{d}^2\boldsymbol{p}}{\mathrm{d}t^2}\right)-\frac{\gamma\boldsymbol{p}}{m^2c^2}\left[\left(\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}\right)^2-\frac{1}{c^2}\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^2\right]\right\}. \tag{1}$$

Exact solution of (1) is clearly intractable. Here we employ a perturbation method. More explicitly, (i) the zeroth-order approximation solves for the electron orbit in the absence of radiation, and (ii) the first-order approximation uses such an orbit to calculate the radiation reaction, which will be substituted in (1) to obtain a more accurate parametrisation of the particle's trajectory. It is obvious that the scheme can be applied iteratively, yielding a progressively better representation of the truth. It would also become evident that the first-order approximation is quite sufficient for dealing with a wide range of field strengths and particle energies.

In the absence of radiation, the equation of motion (1) is

$$\mathrm{d}\boldsymbol{p}/\mathrm{d}t=e\boldsymbol{v}\times\boldsymbol{B}. \tag{2}$$

The solution of (2) yields circular motion in the XY plane (the plane taken to be perpendicular to the magnetic field). More precisely, (2) implies

$$\frac{\mathrm{d}p_x}{\mathrm{d}t}=-m\omega_c v_y \qquad \frac{\mathrm{d}^2p_x}{\mathrm{d}t^2}=-\frac{m\omega_c^2}{\gamma}v_x$$

$$\frac{\mathrm{d}p_y}{\mathrm{d}t}=m\omega_c v_x \qquad \frac{\mathrm{d}^2p_y}{\mathrm{d}t^2}=-\frac{m\omega_c^2}{\gamma}v_y$$

$$\frac{\mathrm{d}p_z}{\mathrm{d}t}=\text{constant} \qquad \frac{\mathrm{d}E}{\mathrm{d}t}=0$$

where $\omega_c=eB/m_e$ is the cyclotron frequency.

Substituting the above radiation reaction terms in (1), and ignoring motion in the z direction, leads to the following differential equations:

$$\frac{\mathrm{d}p_x}{\mathrm{d}t}=-m_e(\omega_c v_y+\alpha\gamma^2 v_x) \qquad \frac{\mathrm{d}p_y}{\mathrm{d}t}=m_e(\omega_c v_x-\alpha\gamma^2 v_y) \tag{3}$$

where $\alpha=2e^2\omega_c^2/3m_ec^3$ and $\gamma=(1-v^2/c^2)^{-1/2}$. Solution of (3) then gives the first-order approximation to the electron orbit.

We proceed by first noting that (3) estimates the energy loss rate to radiation as

$$\frac{\mathrm{d}E}{\mathrm{d}t}=\boldsymbol{v}\cdot\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}=-m_e\alpha\gamma^2\boldsymbol{v}^2=-m_e\alpha c^2(\gamma^2-1)$$

or, equivalently,

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}=-\frac{\alpha\gamma^2v^2}{c^2}=-\alpha(\gamma^2-1). \tag{4}$$

Equation (4) may be solved rather trivially for relativistic energies (i.e. $\gamma\gg1$). We have

$$\gamma(t)=(\alpha t+\beta)^{-1} \tag{5}$$

where $\beta=1/\gamma(t=0)=1/\gamma_0$. We now rewrite (3) by using the relations

$$\frac{\mathrm{d}p_x}{\mathrm{d}t}=m_e\gamma\frac{\mathrm{d}v_x}{\mathrm{d}t}+m_ev_x\frac{\mathrm{d}\gamma}{\mathrm{d}t}$$

etc, and $d\gamma/dt$ from (4):

$$\gamma\frac{dv_x}{dt} = -\omega_c v_y - \alpha v_x \qquad \gamma\frac{dv_y}{dt} = \omega_c v_x - \alpha v_y .$$

In the limit of high γ we may recast the above equation by means of (5):

$$\begin{aligned}\frac{dv_x}{dt} &= -\omega_c(\alpha t+\beta)v_y - \alpha(\alpha t+\beta)v_x \\ \frac{dv_y}{dt} &= \omega_c(\alpha t+\beta)v_x - \alpha(\alpha t+\beta)v_y .\end{aligned} \tag{6}$$

A formal solution to (6) is

$$\begin{aligned}v_x(t) &= c\exp[-\tfrac{1}{2}(\alpha t+\beta)^2]\cos[(\omega_c/2\alpha)(\alpha t+\beta)^2] \\ v_y(t) &= c\exp[-\tfrac{1}{2}(\alpha t+\beta)^2]\sin[(\omega_c/2\alpha)(\alpha t+\beta)^2].\end{aligned}$$

This is a solution of the original differential equation (3) to order $1/\gamma^2$. In fact, with the same degree of accuracy the exponential function may also be written as $[1-(\alpha t+\beta)^2/2]$, so that we have

$$\begin{aligned}v_x(t) &= c[1-\tfrac{1}{2}(\alpha t+\beta)^2]\cos[(\omega_c/2\alpha)(\alpha t+\beta)^2] \\ v_y(t) &= c[1-\tfrac{1}{2}(\alpha t+\beta)^2]\sin[(\omega_c/2\alpha)(\alpha t+\beta)^2].\end{aligned} \tag{7}$$

The initial conditions implicit in (7) are

$$v_x = v_0\cos\phi_0 \qquad v_y = v_0\sin\phi_0 \tag{8}$$

where $v_0^2 = c^2(1-1/\gamma_0^2)$ and $\phi_0 = \omega_c/2\alpha\gamma_0^2$. It is important to recognise ϕ_0/π as the inverse of the fractional energy loss per cycle at $t=0$. The initial conditions on the particle velocity may be altered by introducing a phase term in the argument of the trigonometric functions in (7).

Temporal evolution of the electron position is now obtainable by integration of (7), assuming the initial condition (8). Thus

$$x(t) = c\int_0^t \cos\left(\frac{\omega_c}{2\alpha}(\alpha t'+\beta)^2\right)dt' - \frac{c}{2}\int_0^t (\alpha t'+\beta)^2\cos\left(\frac{\omega_c}{2\alpha}(\alpha t'+\beta)^2\right)dt' \tag{9}$$

and a similar expression for $y(t)$. None of the integrals can be written in closed form, but considerable insight is gained by a change of variables to

$$\phi = \frac{\omega_c(\alpha t+\beta)^2}{2\alpha}$$

in which case (9) becomes

$$\begin{aligned}x(t) &= c(2\alpha\omega_c)^{-1/2}\int_{\phi_0}^{\phi(t)}\left(\psi^{-1/2} - \frac{\alpha}{\omega_c}\psi^{1/2}\right)\cos\psi\, d\psi \\ y(t) &= c(2\alpha\omega_c)^{-1/2}\int_{\phi_0}^{\phi(t)}\left(\psi^{-1/2} - \frac{\alpha}{\omega_c}\psi^{1/2}\right)\sin\psi\, d\psi.\end{aligned} \tag{10}$$

The initial conditions for the solutions (10) are $x=0$, $y=0$ at $t=0$.

Temporal evolution of the instantaneous guiding centre is given by

$$x_0(t) = x(t) - \frac{1}{\omega_c}\gamma(t)v_y(t) \qquad y_0(t) = y(t) + \frac{1}{\omega_c}\gamma(t)v_x(t)$$

where $\gamma(t)$, $\boldsymbol{v}(t)$ and $\boldsymbol{r}(t)$ are assembled from (5), (7) and (10).

3. Orbit characteristics

To examine some gross properties of the decaying electron orbit, we may remove the α/ω_c term in the integrand of (10): this leaves an error of $1/\gamma^2$ which is quite irrelevant. Equation (10) then becomes

$$x(t) = c(2\alpha\omega_c)^{-1/2} \int_{\phi_0}^{\phi(t)} \psi^{-1/2} \cos\psi \, d\psi$$

$$y(t) = c(2\alpha\omega_c)^{-1/2} \int_{\phi_0}^{\phi(t)} \psi^{-1/2} \sin\psi \, d\psi.$$

The integrals are complex error integrals (Gradshteyn and Ryzhik 1965). In fact, the electron trajectory resembles a 'Cornu spiral' well known in diffraction optics. The behaviour in the domains of low and high emission rates are very different and it is best to address them separately.

(I) If in (10) we start with a lower limit of integration $\phi_0 \gg 1$, i.e. if the fractional energy loss per cycle is initially small, then (10) may be replaced by an asymptotic series. More specifically, successive integrations by parts yield the following equations:

$$\begin{aligned} x(t) &= c(2\alpha\omega_c)^{-1/2}\left[\left(\frac{\sin\phi}{\phi^{1/2}} - \frac{\cos\phi}{2\phi^{3/2}} + \ldots\right) - \left(\frac{\sin\phi_0}{\phi_0^{1/2}} - \frac{\cos\phi_0}{2\phi_0^{3/2}} + \ldots\right)\right] \\ y(t) &= -c(2\alpha\omega_c)^{-1/2}\left[\left(\frac{\cos\phi}{\phi^{1/2}} + \frac{\sin\phi}{2\phi^{3/2}} - \ldots\right) - \left(\frac{\cos\phi_0}{\phi_0^{1/2}} + \frac{\sin\phi_0}{2\phi_0^{3/2}} - \ldots\right)\right]. \end{aligned} \tag{11}$$

If we retain only the highest order ($\phi^{-1/2}$) terms in each series, and consider short durations τ around the epoch t, (11) may be simplified to become

$$x(\tau) = x_0 + \frac{c}{\omega_0}\sin(\omega_0\tau + \varepsilon) \qquad y(\tau) = y_0 - \frac{c}{\omega_0}\cos(\omega_0\tau + \varepsilon) \tag{12}$$

where $\omega_0 = \omega_c(\alpha t + \beta) = \omega_c/\gamma$ and $\varepsilon = \omega_c/2\alpha\gamma^2$. The motion now appears instantaneously circular. The constants (x_0, y_0), given by

$$x_0(t) = -\frac{\gamma_0 c}{\omega_c}\sin\phi_0 \qquad y_0 = \frac{\gamma_0 c}{\omega_c}\cos\phi_0 \tag{13}$$

determine the guiding centre position. They are equivalent to the value of the coordinates at $t = 0$, meaning that the guiding centre does not move during radiation.

(II) The situation opposite to (I) is when $\phi_0 \ll 1$ at $t = 0$, i.e. fractional energy loss per cycle is initially large. The electron trajectory is then a complex spiral and the guiding centre drifts rapidly. Most results are unavailable in analytic (closed) form, but numerical computations provide some appreciation of the behaviour (see figures 1 and 2). However, the system settles down within a short fraction of its initial period and the steady state values of various parameters are relatively easy to calculate. We simply require the upper limit of integration in (10) to be large, i.e. $\phi(t) \gg 1$. The technique is to express (10) in the form

$$x(t) = c(2\alpha\omega_c)^{-1/2}\left(\int_{\phi}^{\zeta} \psi^{-1/2}\cos\psi \, d\psi + \int_{\zeta}^{\phi(t)} \psi^{-1/2}\sin\psi \, d\psi\right) \tag{14}$$

and similarly $y(t)$. Here ζ is some suitable intermediate limit such that $\zeta \gg 1$ but $\zeta < \phi(t)$. The first term in (14) is then merely a constant and the second term describes

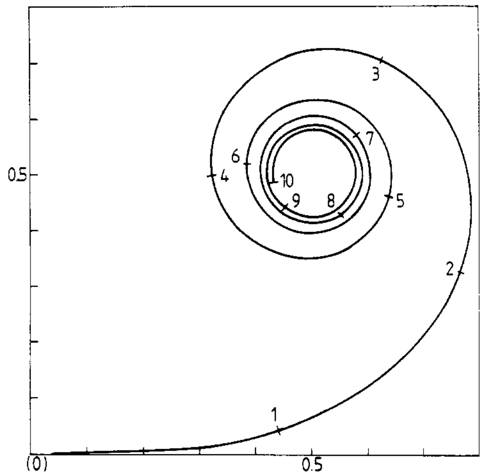

Figure 1. The orbit of an electron of initial Lorentz factor 5×10^6 entering a magnetic field $H=2\times10^6$ G. At time $t=0$ the particle is at the origin (0, 0), the point of entry. Positions marked $1, 2, 3, \ldots, 10$ correspond to times $\omega_0 t = 0.01, 0.02, \ldots, 0.10$ where $\omega_0=\omega_c/\gamma_0$ is the initial angular frequency of rotation. The unit of distance is the metre.

instantaneous circular motion. In fact, (12) still holds with the constants replaced by

$$\left.\begin{matrix} x_0 \\ y_0 \end{matrix}\right\} = c(2\alpha\omega_c)^{-1/2}\int_0^\infty \psi^{-1/2}\left\{\begin{matrix}\cos\psi\\ \sin\psi\end{matrix}\right\} \mathrm{d}\psi = \frac{1.253c}{(2\alpha\omega_c)^{1/2}}. \tag{15}$$

Strictly speaking, the lower limit of integration in (15) should be ϕ_0. But for small ϕ_0 it is a good approximation. This means the relative position of the final guiding centre is independent of electron initial energy, but depends only on the magnetic field via the integral coefficients. Bearing in mind that the guiding centre would have remained in the position given by (13) had there been no radiation damping, (15) indicates a significant particle drift motion caused by radiation.

4. Power spectrum and radiation rate

A key observable consequence of radiation reaction effects is the deviation of the power spectrum from that of conventional synchrotron emission. To calculate this, we consider a scenario similar to that of figures 1 and 2, namely an electron with velocity $\boldsymbol{v}=(v_0, 0)$ entering a region of magnetic field at x, y, $t=0$. The subsequent configuration is given by

$$v_x(t) = c[1-\tfrac{1}{2}(\alpha t+\beta)^2]\cos(\tfrac{1}{2}\alpha\omega_c t^2+\omega_c\beta t) \tag{16a}$$

$$v_y(t) = c[1-\tfrac{1}{2}(\alpha t+\beta)^2]\sin(\tfrac{1}{2}\alpha\omega_c t^2+\omega_c\beta t). \tag{16b}$$

Equation (16) is just a variation of (7) with a phase term added to the trigonometric functions to meet the required initial conditions. For a very brief moment after $t=0$ the electron releases most of its energy in a single pulse and its trajectory is non-circular. The azimuthal angle for $\boldsymbol{v}$ in (16) is, for finite values of it, the inverse of the fractional energy loss per cycle. This means, when the particle begins to turn, it is no longer radiating significantly.

The main concern, therefore, is in the frequency distribution of the very bright pulse which an observer stationed somewhere along the positive x axis would see when

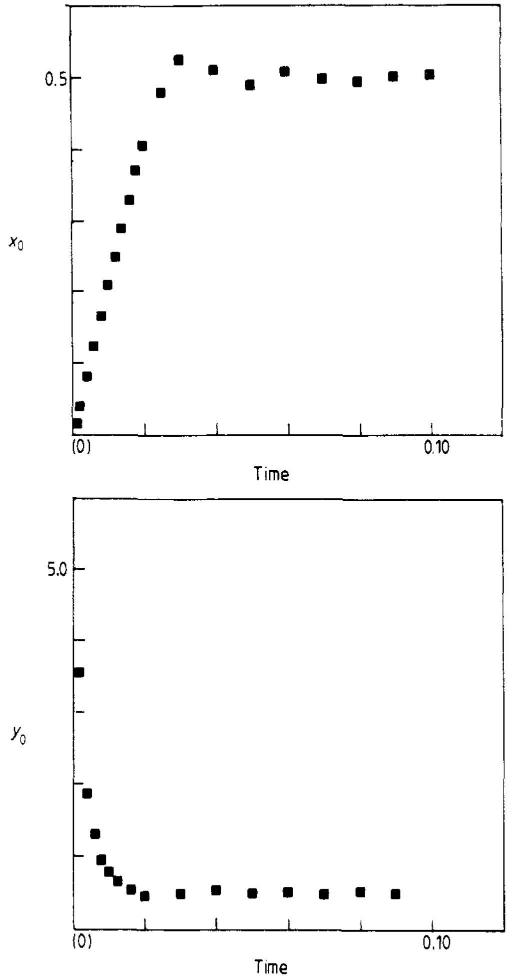

Figure 2. Guiding centre position as a function of $\omega_0 t$ for a condition identical to that described in figure 1. The unit of distance is the metre.

the particle enters the field region. It is then only necessary to consider short time intervals from $t=0$. In addition we concentrate on emission in the orbital plane $\theta=0$. Following the notation in Jackson's treatment of synchrotron radiation (Jackson 1975), the differential power radiated is given by

$$\frac{\mathrm{d}^2 I}{\mathrm{d}\omega\,\mathrm{d}\Omega}=\frac{e^2\omega^2}{4\pi^2 c^3}\left|\int_0^\infty (\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{v})\exp\left[\mathrm{i}\omega\left(t-\frac{\boldsymbol{n}\cdot\boldsymbol{r}}{c}\right)\right]\mathrm{d}t\right|^2 \tag{17}$$

where $\boldsymbol{n}$ is the position vector of the observer. The quantities $\boldsymbol{r}(t)$ and $\boldsymbol{v}(t)$ in (17) are now obtained by an expansion of (16) for small times:

$$\begin{aligned} v_x(t)&=v_0-c\left(\alpha\beta t+\frac{\omega_0^2 t^2}{2}+\frac{\gamma_0\omega_0^2\alpha t^3}{2}+\frac{\gamma_0^2\omega_0^2\alpha^2 t^4}{8}\right)\\ x(t)&=v_0 t-c\left(\frac{\alpha\beta t^2}{2}+\frac{\omega_0^2 t^3}{6}+\frac{\gamma_0\omega_0^2\alpha t^4}{8}+\frac{\gamma_0^2\omega_0^2\alpha^2 t^5}{40}\right)\end{aligned} \tag{18}$$

where $\omega_0 = \omega_c/\gamma_0$, and likewise expressions for $v_y(t)$, $y(t)$. The omitted terms in the expansion of (18) are of order γ_0^2 less than those kept. In fact (18) is valid so long as $\omega_0 t \ll 1$. When γ_0 is large, results are correct for the most important range of frequencies.

We continue along the lines of Jackson and define the variables

$$x = \gamma_0\omega_0 t \qquad \xi = \omega/3\gamma_0^3\omega_0.$$

Combining (17) and (18) we have

$$\frac{d^2I}{d\omega\, d\Omega} = \frac{e^2\omega^2}{4\pi^2\omega_0^2\gamma_0^4 c}\left|\int_0^\infty \left(x+\frac{\mu x^2}{2}\right)\exp[i\xi(\tfrac{3}{2}x+\tfrac{3}{2}\mu x^2+\tfrac{1}{2}x^3+\tfrac{3}{8}\mu x^4+\tfrac{3}{40}\mu^2 x^5)]\, dx\right|^2 \tag{19}$$

where $\mu = \alpha/\omega_0 = \gamma_0\alpha/\omega_c$. The quantity μ is, in fact, the fractional energy loss during pulse crossing. For substantial modification to the spectrum, we require $\mu \simeq 1$. In other words, the electron must assume a non-circular orbit when the pulse transits an observer. Since μ is a critical parameter, it is written in the following form for easy reference:

$$\mu = 1.1\times 10^{-16}\gamma B$$

where B is measured in gauss.

Figures 3 and 4 show power spectra for synchrotron radiation for the cases $\mu = 0.001$ and $\mu = 0.2$, and how they compare with the conventional spectrum which assumes no damping effects. For $\mu = 0.001$ the only noticeable difference is a depletion at low frequencies. For $\mu = 0.2$, the entire spectrum falls below the conventional half-pulse emission curve, and this is because the electron energy decreases appreciably even within a pulse duration.

The total rate of energy loss to synchrotron radiation is obtained by substituting the improved electron orbit (3) and (4) in the formula for radiated power:

$$P = \frac{2e^2}{3m_e^2c^3}\gamma^2\left[\left(\frac{dp}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{dt}\right)^2\right].$$

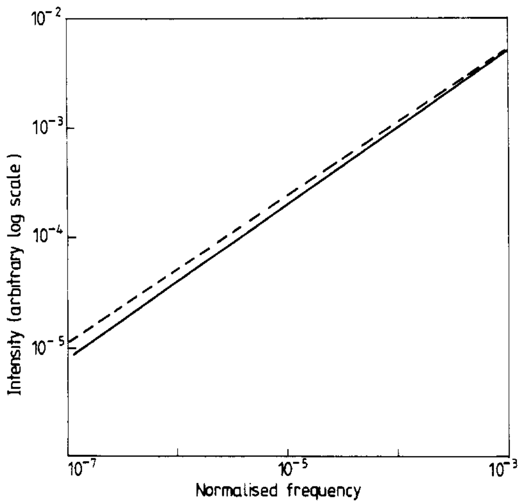

Figure 3. Power spectrum of synchrotron radiation for an electron injected into a region of strong field. Only the initial 'half-pulse' when the particle velocity vector crosses the observer's line of sight (positive x axis) is considered. The lower curve corresponds to a 0.1% energy loss in the event. No significant differences in the spectra are noted for $\xi \gtrsim 10^{-3}$ (ξ is the normalised frequency, see text).

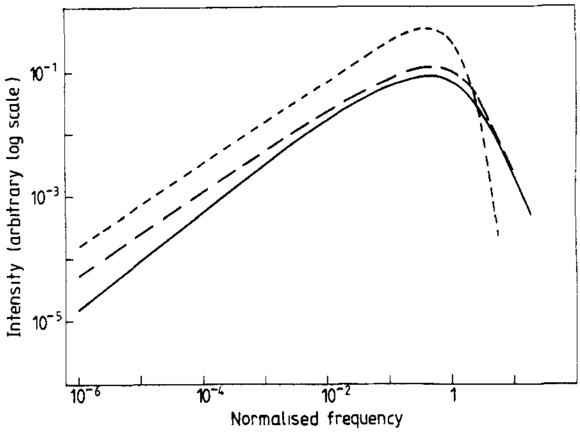

Figure 4. As in figure 3, with the exception that 20% of the initial electron energy is lost to radiation within the initial pulse transit time. Also included in this diagram is the steady-state 'full-pulse' spectrum, computed on the assumption that, in the absence of radiation damping, the particle returns to the observer, i.e. the synchrotron beam makes a complete sweep. ---, full pulse, no damping; ———, half-pulse, no damping; —— half-pulse, $\mu = 0.2$.

This gives the result

$$P = m_e \alpha c^2(\gamma^2 - 1)(1 + \mu^2).$$

When compared with (4), it is evident that first-order perturbation estimates an increase in emission rate by a factor $(1 + \mu^2)$.

5. Limitations of the theory

The calculated orbit and radiation characteristics are subject to further modifications in situations of high-energy loss. It is therefore important to investigate the criterion of validity of first-order perturbation theory. Here it is simply stated, without details of proof, that higher-order corrections to the electron orbit carry terms of order μ^n, where $n \geq 2$, in the differential equations for $\mathrm{d}p/\mathrm{d}t$. The additional terms introduced in (17)-(19) then fall short of the existing terms by a factor $\geq 1/\mu^2$. Hence for small μ the results presented are reliable. As μ approaches unity, the successive high orders take their place. For $\mu > 1$, the perturbation method cannot be used.

Acknowledgments

I am indebted to Richard Drachman of NASA/GSFC and Hugh Jones at ICST for helpful discussions and ideas.

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